VECTORS

Displacement and Positions of Vectors

The Concept of a Vector Quantity

Explain the concept of a vector quantity

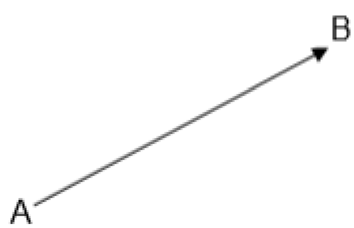
A vector - is a physical quantity which has both magnitude and direction.

The Difference Between Displacement and Position Vectors

Distinguish between displacement and position vectors

If an object moves from point A to another point say B, there is a displacement

$\overrightarrow{AB}$ which is the distance from A to B.



From the above figure $\overrightarrow{AB}$ is the Vector since it has magnitude as well as direction.

There are many Vector quantities, some of which are: displacement, velocity, acceleration, force, momentum, electric field and magnetic field.

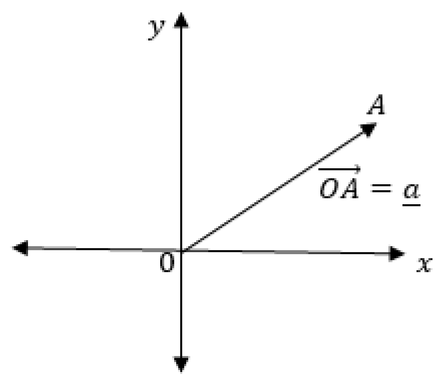
Other physical quantities have only magnitude, these quantities are called **Scalars**.

For example distance, speed, pressure, time and temperature

Naming of Vectors:

Normally vectors are named by either two capital letters with an arrow above e.g.

$\overrightarrow{OA}$, $\overrightarrow{AB}$. etc. also a single capital letter or small letter in bold print. E.g. and sometimes a single small letters with a bar below.

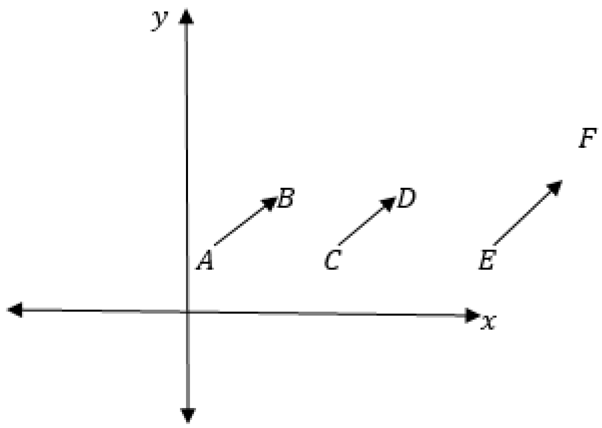


Note that in the notation $\overrightarrow{OA}$, the arrow indicates the direction of the vector. Therefore the vector $\overrightarrow{OA}$ has the initial point at O and end point at A, while the vector $\overrightarrow{AO}$ has its initial point at A and end point at O, thus $\overrightarrow{AO} \neq \overrightarrow{OA}$ because they have different directions though the same magnitude.

Equivalent Vectors:

Suppose $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are two vectors such that the length (distance) from A to B is the same as that from C to D and the direction from A to B is the same as that from C to D , then we say $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent vectors.

Therefore two or more vectors are said to be equivalent if and only if they have same magnitude and direction.

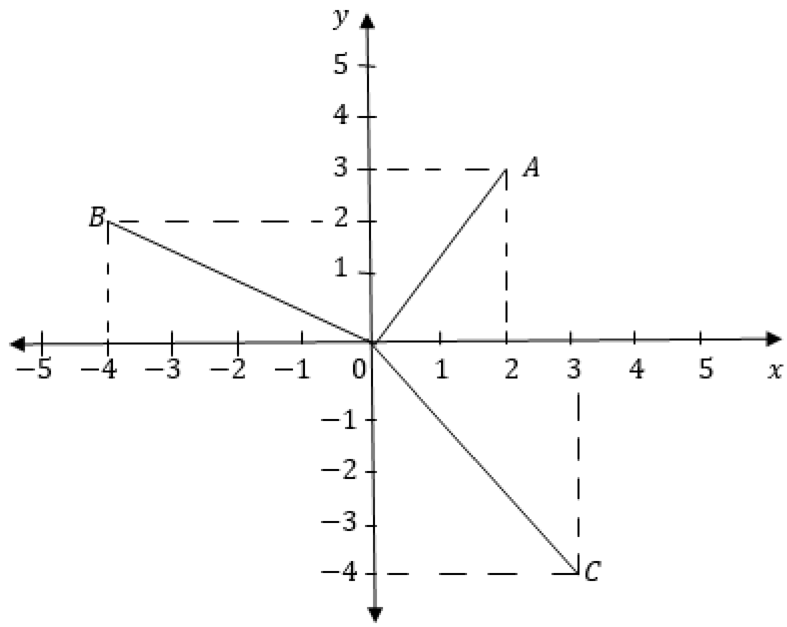


From the figure above, $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are equivalent vectors.
Normally we write $\overrightarrow{AB} \equiv \overrightarrow{CD}$ to mean $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$, thus if $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are equivalent, then we write $\overrightarrow{AB} \equiv \overrightarrow{CD} \equiv \overrightarrow{EF}$

Position Vectors;

In the x —plane all vectors with initial points at the origin and their end points elsewhere are called position vectors. Position vectors are named by the coordinates of their end points.

Consider the following diagram.



From Fig. 3 above, $\overrightarrow{OA}$ = (2, 3), $\overrightarrow{OB}$ = (-4, 2) and $\overrightarrow{OC}$ = (3, -4) are position vectors of points A, B, and C respectively.

Components of position vectors:

Any position vector $\overrightarrow{OP}$ = (x, y) can be resolved into two components, a horizontal component and vertical component. Thus the components of $\overrightarrow{OP}$ = (x, y) are (x, 0) and (0, y) where (x, 0) is the horizontal component and (0, y) is the vertical component.

Example 1

Write the position vectors of the following points: (a) A (1,-1), (b) B (-4,-3)

(c) C= (u, v) where U and V are any real numbers and give their horizontal and vertically components

Solution:

(a) $\overrightarrow{OA} = (1,-1)$

Horizontal component = (1, 0)

Vertical component = (0,-1)

(b) $\overrightarrow{OB} = (-4,-3)$

Horizontal component = (-4, 0)

Vertical component = (0,-3)

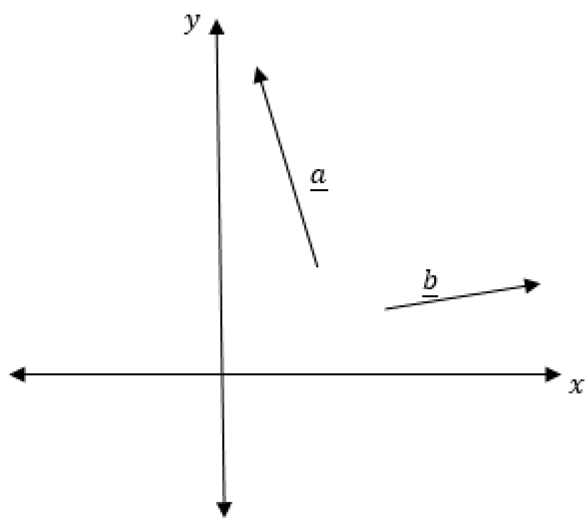
(c) $\overrightarrow{OC} = (u, v)$

Horizontal component = (u, 0)

Vertically component = (0, v)

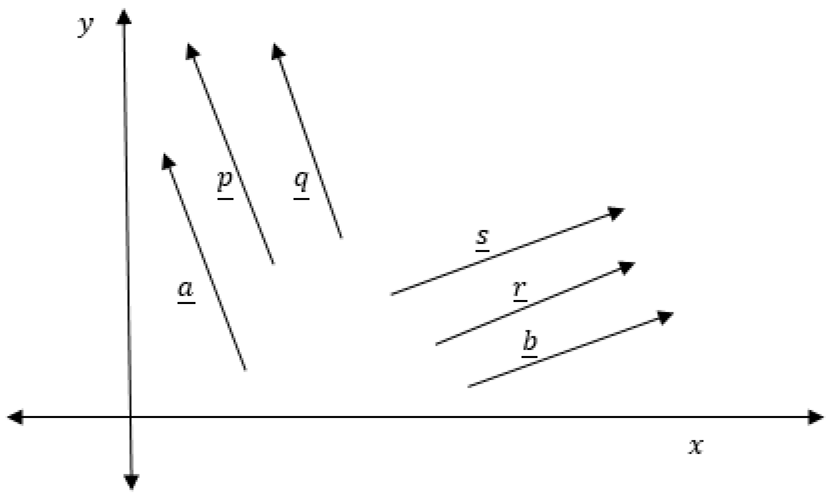
Example 2

For each of vectors a and b shown in figure below draw a pair of equivalent vectors



Solution:

The following figure shows the vectors **a** and **b** and their respective pairs of equivalent vectors



Any Vector into I and J Components

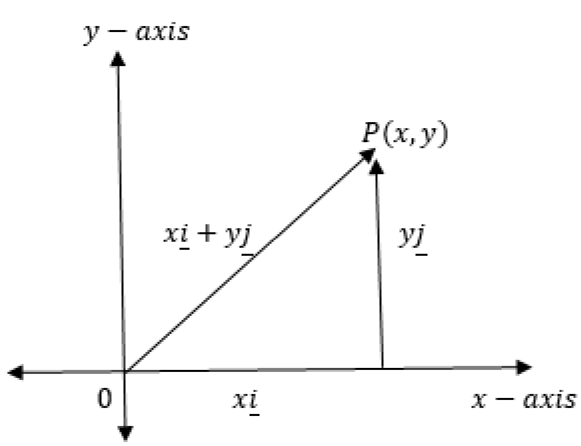
Resolving any vector into I and J components

The unit Vectors i and j.

Definition*:* A unit vector is a position vector of unit length in the positive direction of x axis or y axis in the xy—plane.

The letters iand jare used to represent unit vectors in the X axis and y – axis respectively.

Consider the following sketch,



We can write i and j in terms of position vectors as i= (1, 0) and j = (0, 1), from the figure above, $\overrightarrow{OP} = (x, y)$

Now $\overrightarrow{OP}$ can be resolved into $\overrightarrow{OP} = (x, 0) + (0, y)$ and factorized into $\overrightarrow{OP} = x\,(1, 0) + y\,(0, 1)$ But i = (1, 0) and j = (0, 1)

So $\overrightarrow{OP} = x\,(1, 0) + y\,(0, 1)$

Example 3

Write the following vectors in terms of i and j vectors:

(a) **a**= (-3,-4) , (b) **b** = (-5,5)

(c) **c** = (3,2) and (d) **d** = (u, v)

Solution

(a) **a**=(-3,-4) = (-3,0) + (0,-4)

= -3 (1, 0) + -4(0, 1)

=-3i+-4j

= -3i – 4j

(b) **b**= (-5,5) = -5i + 5j

(c) **c** = (3,2) = 3i + 2j

(d) **d** = (u, v) = ui + vj

Example 4

Write the following vectors as position vectors.

(a) S =-8i (b) U = 7, j (c) V= $^2/_3$i + $^1/_6$ j (d) D = ui + vj

Solution

(a) S = -8i

S = -8i + 0j

= - 8 (1, 0) + 0 (0, 1)

= (-8, 0) + (0, 0)

= (-8, 0)

(b) U = 7j = 0i + 7j

U = 0(1, 0) + 7 (0, i)

= (0, 0) + (0, 7)

= (0, 7)

(c) V= $^2/_3$ (1, 0) + $^1/_6$ (0, 1)

= $^2/_3$ (1, 0) + $^1/_6$ (0, 1)

= ($^2/_3$, 0) + (0, $^1/_6$)

= ($^2/_3$, $^1/_6$)

(d) D= ui + vj

= u (1, 0) + (0, v)

= (u, 0) + (0, v)

= (u, v)

Magnitude and Direction of a Vector

The Magnitude and Direction of a Vector

Calculate the magnitude and direction of a vector

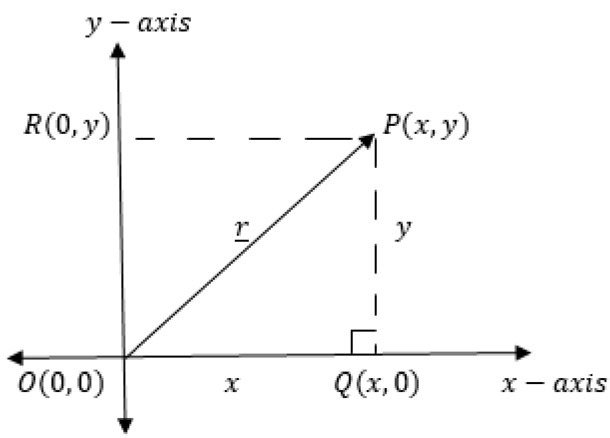
Magnitude (Modules) of a Vector

Definition: The magnitude / modules of a vector is the size of a vector, it is a scalar quantity that expresses the size of a vector regardless of its direction.

Finding the magnitude of given vector.

Normally the magnitude of a given vector is calculated by using the distance formula which is based on Pythagoras theorem.

Let $\overrightarrow{OP}$ = (x, y) be the position vector on the xy –plane.



From the figure above, **r** = $\overrightarrow{OP}$ = (x, y)

Using Pythagoras theorem

$(\mathbf{r})^2 = (\overrightarrow{op})^2 = (\overline{OQ})^2 + (\overline{PQ})^2 = x^2 + y^2$

Since OQ=x and PQ =y.

So $|\mathbf{r}| = |\overline{OP}| = \ x^2 + y^2$

Now if **r** =(x, y), then its magnitude is demoted by I **r** I which is given by

$$|\mathbf{r}| = \sqrt{x2 + y2}$$

Example 5

Calculate the magnitude of the position vector **v**=(- 3 , 4)

Solution

v= (- 3, 4)

l **v** l = $\sqrt{(-3)2 + 42}$ = $\sqrt{9 + 16}$ = $\sqrt{25}$ =5

Therefore the magnitude of **v** is l **v** l = 5

What is the magnitude of the vector U if U = 4i – 5j?

Solution

Given that U = 4i – 5j

The vector U = 4i – 5j can be written as a position vector as U = (4,-5)

So l**u**l = $\sqrt{42 + (-5)2}$ = $\sqrt{16 + 25}$ = $\sqrt{41}$,

So l **u** l =$\sqrt{41}$.

Unit Vectors:

Definition: A unit of Vector is any vector whose magnitude or modulus is one Unit.

Now if U is any Vector, then the Unit Vector in the direction of U is given by $\frac{U}{\text{IUI}}$ and it is denoted by $\widehat{U}$

Example 6

Find a Unit Vector in the direction of Vector U = (12, 5)

Solution

$$\underline{\hat{U}} = \frac{\underline{U}}{|\underline{U}|}$$

But $\underline{U} = (12, 5)$ and $|\underline{U}| = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$

Then

$$\underline{\hat{U}} = \frac{\underline{U}}{|\underline{U}|} = \frac{(12, 5)}{13}$$

$$\underline{\hat{U}} = \left(\frac{12}{13}, \frac{5}{13}\right)$$

To check out that $\underline{\hat{U}} = \left(\frac{12}{13}, \frac{5}{13}\right)$ is a Unit Vector, you need to find its magnitude

$$|\underline{\hat{U}}| = \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2} = \sqrt{\frac{144}{169} + \frac{25}{169}} = \sqrt{\frac{169}{169}} = 1$$

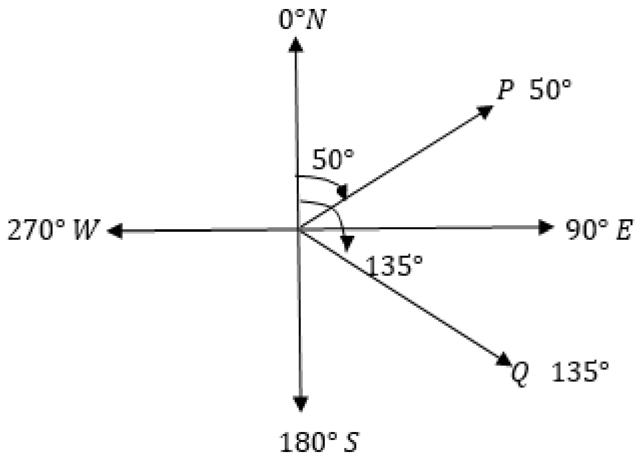
Direction of a vector:

The direction of a Vector may be given by using either bearings or direction Cosines.

(a) **By Bearings**:

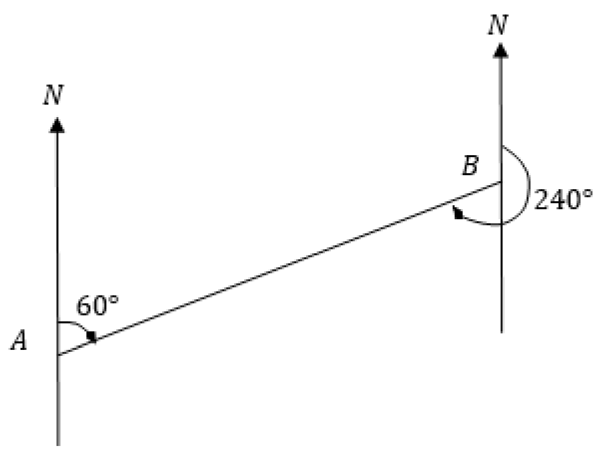
Bearings are angles from a fixed direction in order to locate the interested places on the earth's surface.

Reading bearings: There are two method used to read bearings, in the first method all angles are measured with reference from the North direction only where by the North is taken as 000^0, the east 090^0, the South is 180^0 and West 270^0



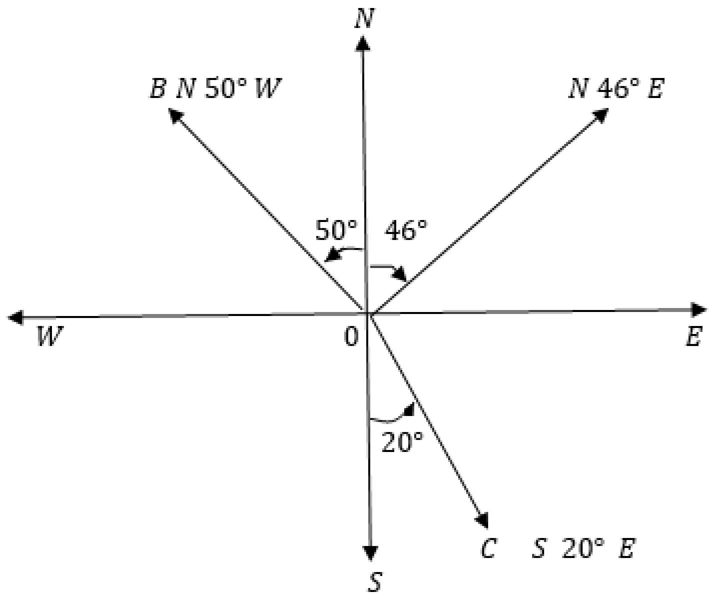
From the figure above, point P is located at a bearing of 050^0, while Q is located at a bearing of 135^0.

Commonly the bearing of point B from point A is measured from the north direction at point A to the line joining AB and that of A from B is measured from the North direction at point B to the line joining BA.



From the figure above the bearing of B from A, is 060^0 while that of A from B is 240^0 In the second method two directions are used as reference directions, these are North and south.

In this method the location of places is found by reading an acute angle from the north eastwards or westwards and from the south eastwards or westwards.



From figure above, the direction of point A from O is N 46^0 E , that of B is N50^0W while the direction from of C is S20^0E.

Example 7

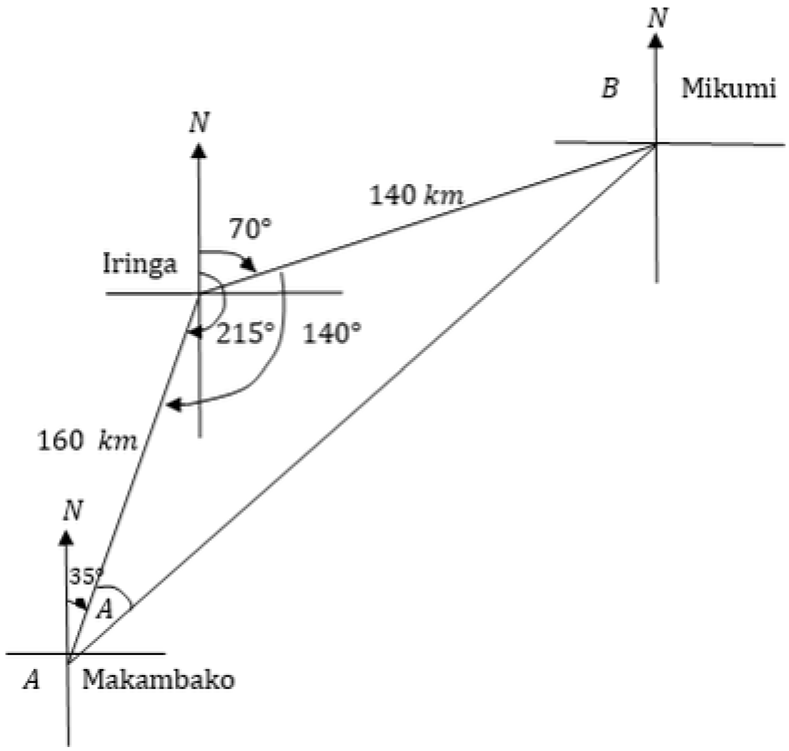
Mikumi is 140km at a bearing of 070^0f from Iringa. Makambako is 160km at bearing of 215^0 from Iringa. Sketch the position of these towns relative to each other, hence calculate the magnitude and direction of the displacement from Makambako to Mikumi.

Solution

Let 1cm represent 20km, and let $\overrightarrow{AB}$ be the displacement from Makambako to Mikumi.

The following sketch describes the location of these two places relative to each other, since $\overrightarrow{AB}$ Is the displacement from Makambako to Mikumi, then the distance between A and B is the distance between the two places.
This is to say point A stands for Makambako and B for Mikumi.

Sketch



Using the cosine rule

$$\left|\overrightarrow{AB}\right|^2 = 160^2 + 140^2 - 2(160)(140)\cos 145°$$
$$= 25600 + 19600 - 44800\cos 35°$$
$$= 45200 + 36700.16$$
$$= 81\,900.16$$

$$\overrightarrow{AB} = \sqrt{81\,900.16} \approx 286\ km$$

The displacement from Makambakoto Mikumiis 12 286kmBy sine rule

$$\frac{\sin A}{140°} = \frac{\sin 145°}{286}$$

$$\sin A = \frac{140 \text{ x} \sin 145°}{286}$$

$$\sin A = \frac{140 \text{ x} \sin 35°}{286} = \frac{140 \text{ x } 0.5736}{286}$$

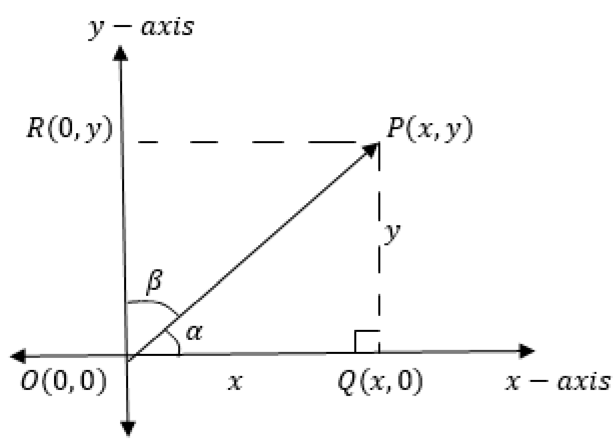
$$\sin A = 0.280$$
$$A = 16.3°$$

The bearing of Mikumi from Makambako is $16.3^0 + 35^0 = 051.3^0$ or $N51.3^0$ W

Alternatively by using the scale AB is approximately 14.3 cm Therefore AB = 14.3x 20 km = 286km and the bearing is obtained a protractor which is about $N51^0E$

(b) **Direction cosines**

Definition: If $\overrightarrow{OP} = (x, y)$ makes angles α and β with the positive directions of x and y axes respectively, then the cosines of α and β are the direction cosines of the vector O P



From the figure above, $P\hat{O}R = O\hat{P}Q = \beta$ and $P\hat{O}Q = O\hat{P}R = \alpha$, also $OP = |\overrightarrow{OP}|$
Therefore

$$\boxed{\cos\alpha = \frac{x}{|\overrightarrow{OP}|} \quad \text{and} \quad \cos\beta = \frac{y}{|\overrightarrow{OP}|}}$$

Where Cos A and Cos B are the direction cosines of OP

Example 8

If **a** = 6i + 8j find the direction cosine of *a* and hence find the angle made by *a* with the positive x – axis.

Solution

Given **a** = 6i + 8j

(a) $|\underline{a}| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$

Let α and β be the angles made by *a* with the positive x and y axes respectively.

Then

$$\cos\alpha = \frac{x}{|a|} = \frac{6}{10} = \frac{3}{10}$$

$$\cos\alpha = \frac{3}{5}$$

$$\cos\beta = \frac{y}{|a|} = \frac{8}{10} = \frac{4}{5}$$

$$\cos\beta = \frac{4}{5}$$

The direction cosines of *a* are $^3/_5$ and $^4/_5$

To find α we consider the direction cosine of x axis and read the angle corresponding to it from mathematical tables.

Now Cos $\alpha = {}^3/_5$, $\alpha = 53.10^0$

Exercise 1

1. Find the magnitude of

A = -12i – 5j

2. Calculate the direction cosines of **a** + **b** if **a** = 3i + 4j and **b** = 7i

3. Let P = 3i + 4j, find the value of (a) $\underline{P} + \frac{\underline{P}}{|\underline{P}|}$

(b) The modulus of $\underline{P} + \frac{\underline{P}}{|\underline{P}|}$

4. Find the value of m if
W = $^2/_3$ j+ m j is a Unit vector.

Sum and Difference of Vectors

The Sum of Two or More Vectors

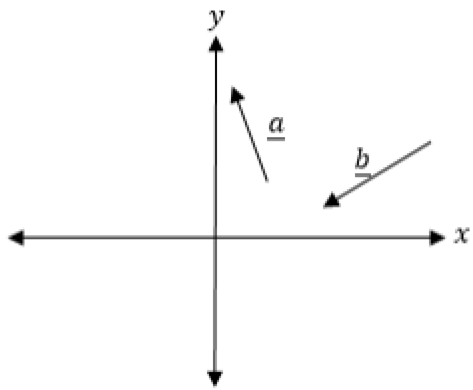
Find the sum of two or more vectors

Addition of vectors

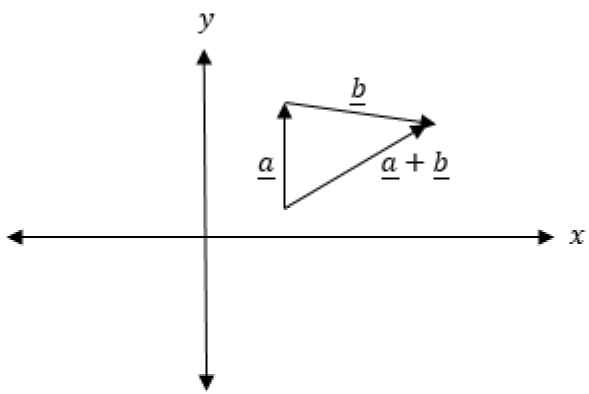
The sum of any two or more vectors is called the **resultant** of the given vectors. The sum of vectors is governed by triangle, parallelogram and polygon laws of vector addition.

(**1**) Triangle law of vector Addition

Adding two vectors involves joining two vectors such that the initial point of the second vector is the end point of first vector and the resultant is obtained by completing the triangle with the vector whose initial point is the initial point of the first vector and whose end points the end point of the second vector.



From the figure above **a** + **b** is the resultant of vectors **a** and **b** as shown below

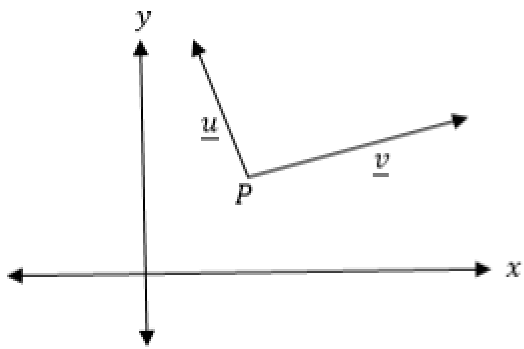


(2) The parallelogram law

When two vectors have a common initial point say P, then their resultant is obtained by completing a parallelogram, where the two vectors are the sides of the diagonal through P and with initial point at P

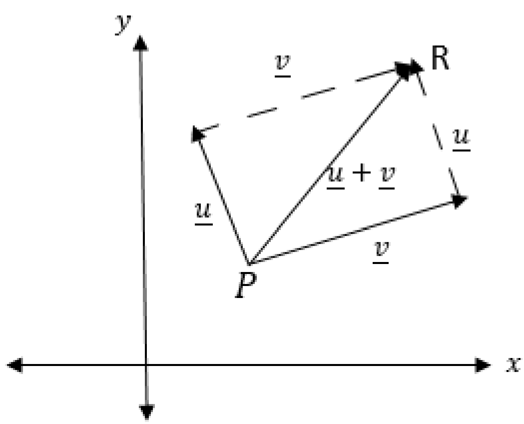
Example 9

Find the resultant of vectors**u**and**v**in the following figure.



Solution

To get the resultant of vectors **u** and **v**, you need to complete the parallelogram as shown in the following figure



From the figure above, the result of **u** and **v** is PR = **U + V** = U + V

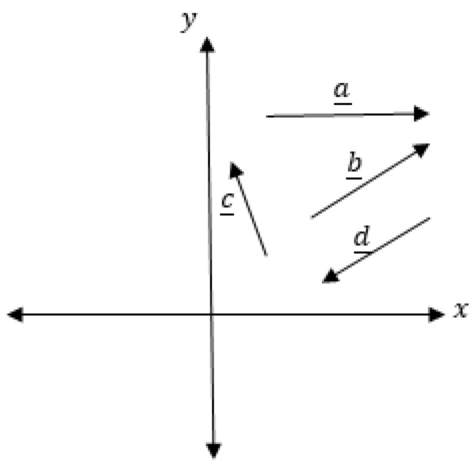
Note that by parallelogram law of vector addition, commutative property is verified.

Polygon law of vector addition*:*

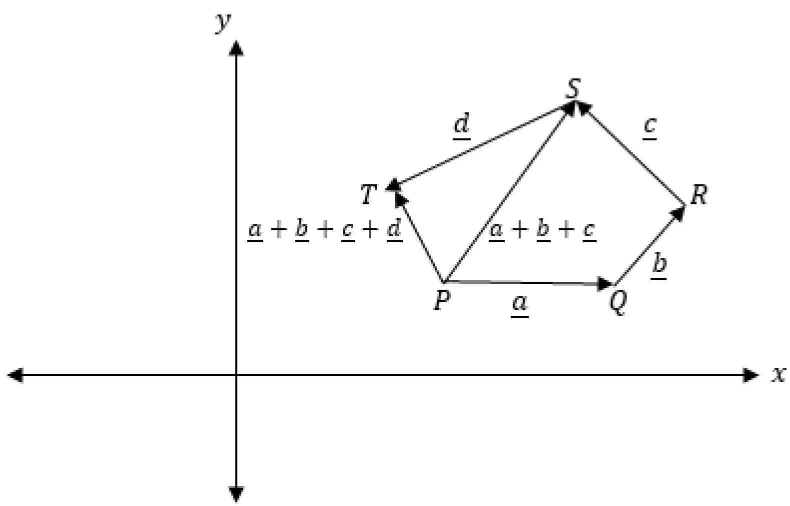
If you want to add more than two vectors, you join the end point to the initial point of the vectors one after another and the resultant is the vector joining the initial point of the first vector to the end point of the last vector

Example 10

Find the resultant of vectors a, b, c and d as shown in the figure below.



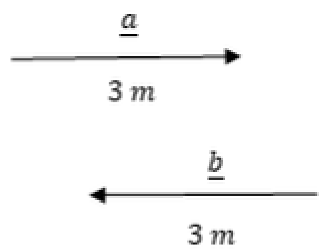
Solution



In the figure above P is the initial point of **a, b** has been joined to **a** at point Q and **c** is joined to **b** at R, while **d** is joined to **c** at point S and PT = **a + b + c + d** which is the resultant of the four vectors.

Opposite vectors

Two vectors are said to be opposite to each other if they have the same magnitude but different directions



From the figure above **a** and **b** have the same magnitude (3m) but opposite direction.

So **a** and **b** are opposite vectors.

Opposite vectors have zero resultant that is if **a** and **b** are opposite vectors, then

$$\boxed{\underline{a} + \underline{b} = 0}$$

Example 11

Find the vector **p** opposite to the vector **r** = 6i – 2j

Solution

Let **p** = ai + bj

Since **p** and **r** are opposite to each other, then **p** + **r** = 0

So (ai + bj) + (6i – 2j) = 0
(ai + 6i) + (6j + 2j) = 0i + 0j = 0
ai + 6i = 0i ……… (1)
bj – 2j = 0j……….(2)
a + 6 = 0 which means a = -6
b-2 = 0 meaning that b=2
but **p**= ai + bj
p= -6i + 2j
Therefore **p** = -6i + 2j

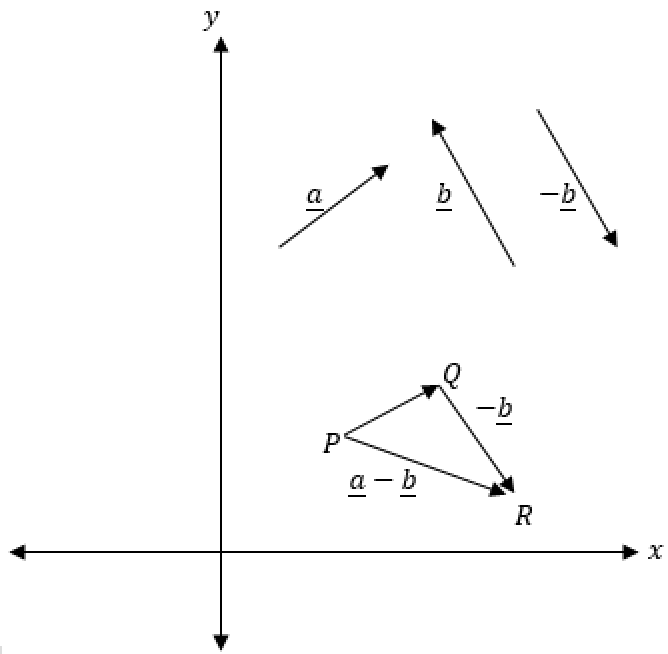
The Difference of Vectors

Find the difference of vectors

Normally when subtracting one vector from another the result obtained is the same as that of addition but to the opposite of the other vector.

Therefore the different of two vector is also the resultant vector

Consider the following figure



From the figure above

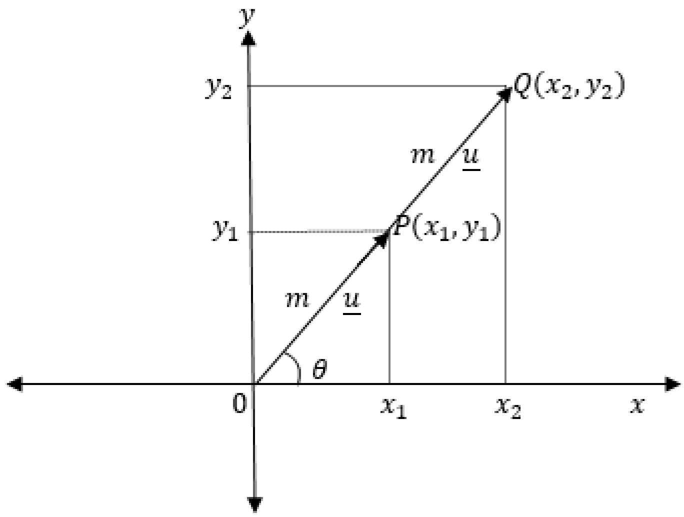
$$\text{Resultant vector} = \overrightarrow{PR} = \underline{a} - \underline{b}$$

Multiplication of a Vector by a Scalar

A Vector by a Scalar

Multiply a vector by a scalar

If a vector U has a magnitude m units and makes an angleθwith a positive x axis, then doubling the magnitude of U gives a vector with magnitude 2m.



From the figure above, $\overrightarrow{OP} = \underline{u}$ and $\overrightarrow{PQ} = \underline{u}$, So $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = \underline{u} + \underline{u} = 2\underline{u}$

Also $|\underline{u}| = m$ which implies that

$$|\underline{u} + \underline{u}| = m + m = 2m$$

Generally if U = (u_1, u_2) and t is any non zero real number while (u_1, u_2) are also real numbers, then

$$t\underline{u} = t(u_1, u_2)$$
$$= (tu_1, tu_2)$$

It follows therefore that the vector (tu_1, tu_2) is a scalar multiple of vector (u_1, u_2).

Similarly if U = u_2i + u_2j,

Then

$$\boxed{t\underline{u} = tu_1\, \underline{i} + tu_2\, \underline{j}}$$

Example 12

If **a** = 3i + 3j and **b** = 5i + 4j

Find – 5**a** + 3**b**

Solution

-5**a** + **3b** = -5 (3i + 3j) + 3 (5i + 4j)

Since **a** = 3i + 3j and **b** = (5i + 4j)

= - 5a + 3b = (-15i + -15j) + (15i + 12j)
= (-15i + 15i) + -15j + 12j)
= 0i + -3j
= - 3j
Therefore-5a + 3b = -3j

Example 13

Given that **p** = (8, 6) and **q** = (7, 9). Find 9p – 8**q**

Solution

Given **p** = (8, 6) and **q**= (7, 9)
9p – 8**q** = 9 (8, 6) – 8 (7, 9)
= (9x8, 9x6) – (8x7, 8x9)
= (72, 54) – (56, 72)
= (72-56), (54-72) = (16, -18)
Therefore 9**p** – 8**q** = (16, -18),

Application of Vectors

Vectors in Solving Simple Problems on Velocities, Displacements and Forces

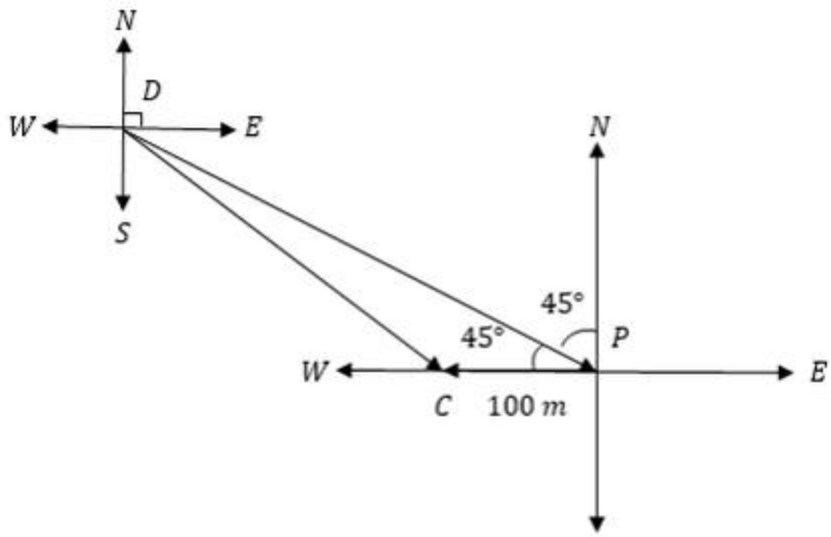
Apply vectors in solving simple problems on velocities, displacements and forces

Vector knowledge is applicable in solving many practical problems as in the following examples.

A student walks 40 m in the direction S 45^0 E from the dormitory to the parade ground and then he walks 100m due east to his classroom. Find his displacement from dormitory to the classroom.

Solution

Consider the following figure describing the displacement which joins the dormitory D. parade ground P and Classroom C.



From the figure above the resultant is DC. By cosine rule

$$\left|\overrightarrow{DC}\right|^2 = 400^2 + 100^2 - 2(400)(100)\cos 45°$$
$$= 160\,000 + 10\,000 - 80\,000\cos 35°$$
$$= 113\,440$$

$$\overrightarrow{DC} = \sqrt{113\,440} \approx 336.8\ m$$

Let $C\widehat{D}P = \theta$, then by sine rule

$$\frac{\sin\theta}{100} = \frac{\sin 45°}{336.8}$$

$$\sin\theta = \frac{100 \text{ x} \sin 45°}{336.8}$$

$$\sin\theta = \frac{100 \text{ x } 0.707}{336.8}$$

$$\theta = 12°$$

Then the bearing is $S(45° - 12°)\,E = S\ 33°\ E$

∴The boys' displacement from the dormitory to the classroom is 336.8 m at a bearing of $S\ 33°\ E$

Example 14

Three forces $F_1 = (3,4)$, $F_2 = (5,-2)$ and $F_3 = (4,3)$ measured in Newtons act at point O (0,0)

a. Determine the magnitude and direction of their resultant.

b. Calculate the magnitude and direction of the opposite of the resultant force.

Solution

(a) Let F be the resultant force

$F = F_1 + F_2 + F_3$

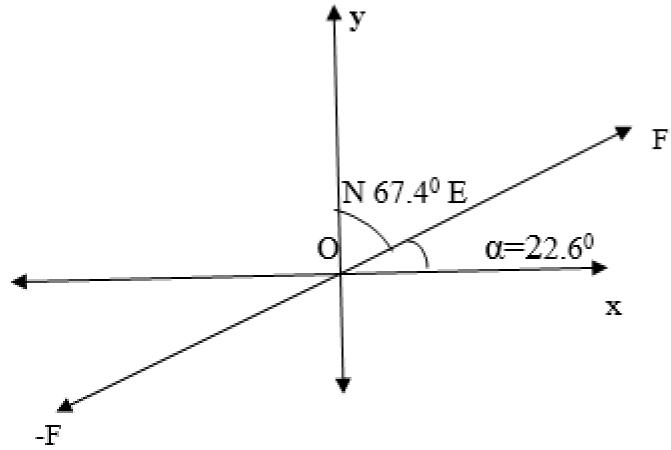
$= (3, 4) + (5,-2) + (4, 3)$

$= (12, 5)$

$|F| = 13N$

Also $\cos\alpha = \frac{12}{13} = 0.92307$

$\alpha = 22.6^0$



From the figure above, the bearing of F is N 67.4^0 E

Therefore the resultant force is 13 N at the bearing of N 67. 4^0 E

(b) Let the force opposite to F be F_o, then Fo = -F = - (12, 5) = (-12, -5)

Fo= 13N and its bearing is $(67.4^0+180^0) = 247.4^0$

So the magnitude and direction of the force opposite to the resultant force is 13N and S67.4^0W respectively..

Exercise 2

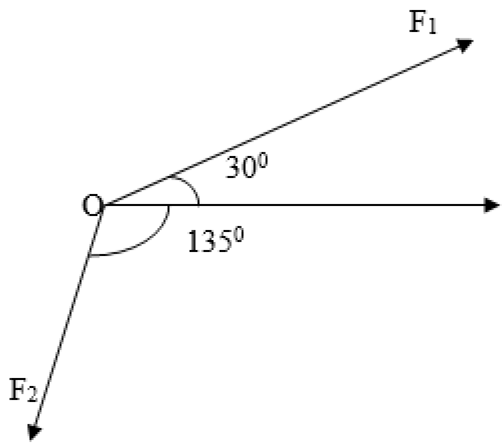
1. Given that U = (3, -4), V= (-4, 3) and W = (1, 1), calculate.

a. The resultant of U + V + W

b. The magnitude and direction of the resultant calculated in part (a) above.

2. A boat moves with a velocity of 10km/h upstream against a downstream current of 10km/h. Calculate the velocity of the boat when moving down steam.

3. Two forces acting at a point O makes angles of 30^0 and 135^0 with their resultant having magnitude 20N as shown in the diagram below.



4. Calculate the magnitude and direction of the resultant of the velocities V_1=5i + 9j,V_2 = 4i + 6j and V_3 = 4i – 3j where i and j are unit vectors of magnitude 1m/s in the positive directions of the x and y axis respectively.

- READ TOPIC 7: Matrices And Transformations