FUNCTIONS

Normally relation deals with matching of elements from the first set called DOMAIN with the element of the second set called RANGE.

Definitions:

A function is a relation with a property that for each element in the domain there is only one corresponding element in the range or co- domain

Therefore functions are relations but not all relations are functions

Representation of a Function

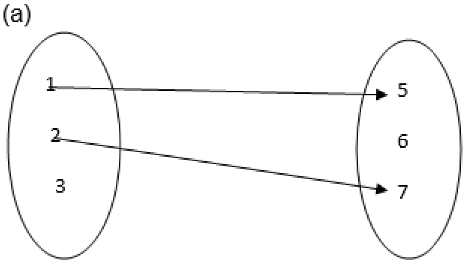
The Concept of a Functions Pictorially

Explain the concept of a functions pictorially

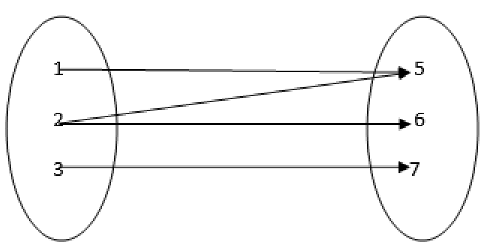
Example 1

Which of the following relation are functions?

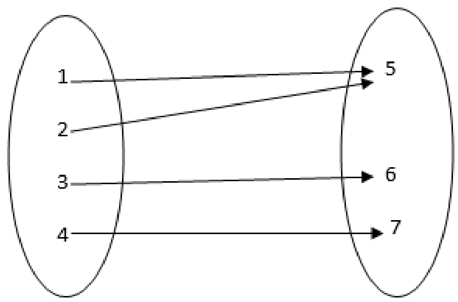
(a)



(b)



(c)



Solution

a. It is not a function since 3 and 6 remain unmapped.

b. It is not a function because 2 has two images (5 and 6)

c. It is a function because each of 1, 2, 3 and 4 is connected to exactly one of 5, 6 or 7.

Functions

Identify functions

TESTING FOR FUNCTIONS;

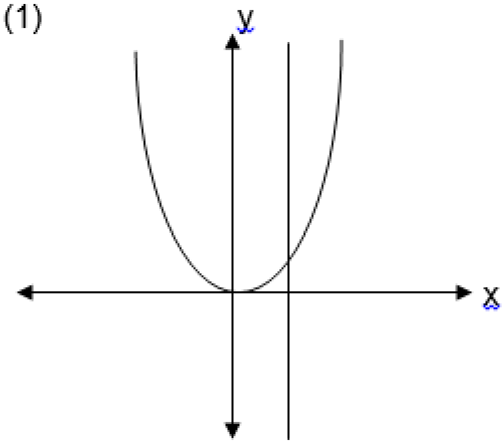
If a line parallel to the y-axis is drawn and it passes through two or more points on the graph of the relation then the relation is not a function.

If it passes through only one point then the relation is a function

Example 2

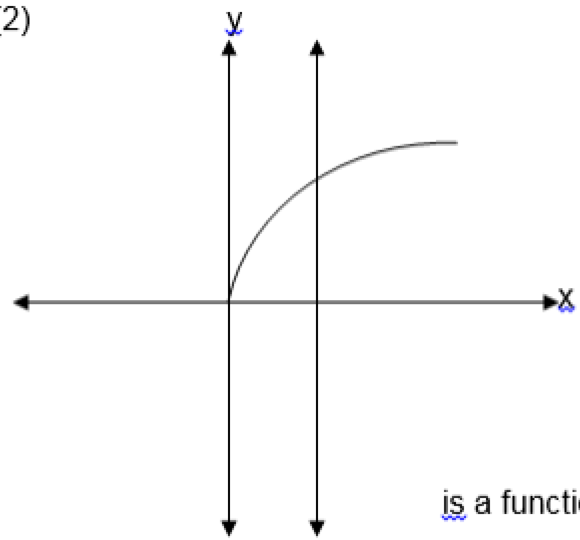
Identify each of the following graphs as functions or not.

(1)



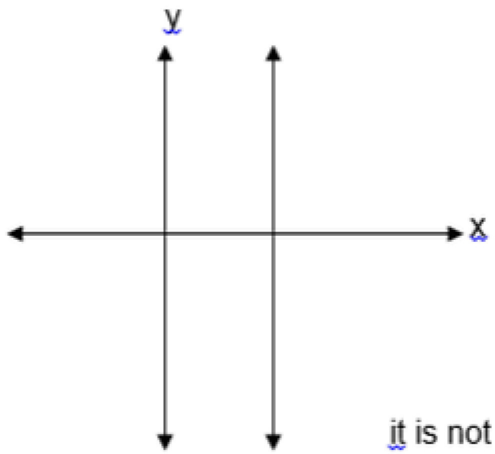
is a function.

(2)



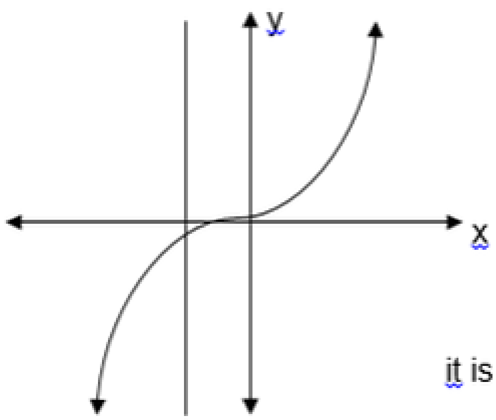
is a function.

(3)



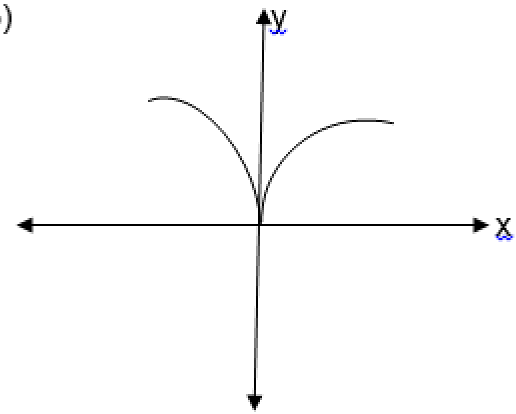
it is not a function.

(4)

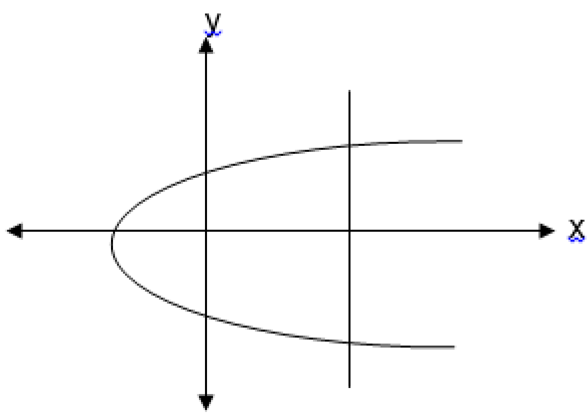


it is a function

(5)



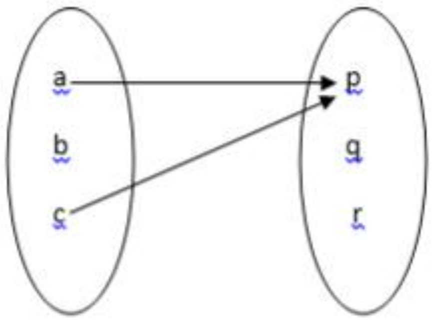
(6)



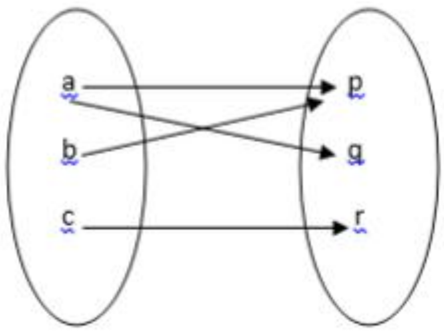
Exercise 1

1. Which of the following relations are functions?

(a)



(b)



2. Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

and B ={ 2, 3, 5, 7 }

Draw an arrow diagram to illustrate the relation “ is a multiple of ‘ is it a function ? why?

3. let A = {1,-1 ,2,-2} and

B = {1, 2, 3, 4 } which of the following relations are functions ?

a. $\{ (x , y) : x < y \}$

b. $\{ (x , y) : x > y\}$

c. $\{ (x , y) : y = x^2\}$

Domain and Range of a Function

The Domain of a Function

State the domain of a function

If y = f (x),that is y is a function of x ,then domain is a set of x values that satisfy the equation y = f (x).

The Range of Function

State the range of function

If y = f (x),that is y is a function of x , then therange is a set of y value satisfying the equation y = f (x).

Example 3

1. Let f(x) = 3x – 5 for all value of x such that -2 £ x £ 3 find its range

Solution

f (x) = y = 3x -5

When x = -2

f(-2) = y = 3x(-2)-5 = -11 , so (x,y)=(-2,-11)

f(3) =y= 3x3-5 = 4, so when x = 3 , y = 4

Therefore y is found in between – 11 and 4

Range ={ y**:** - 11 £ y £ 4}

Example 4

If f (x) = x^2 – 3, state the domain and range of f (x)

Solutions;

Domain = all real numbers

Range:

f(x) = y = $x^2 - 3$

Make x the subject

$y + 3 = x^2$

$\sqrt{y + 3} = x$

Since there is no square root of negative number(s) then y + 3 must be greater than or equal to zero.

i.e $y + 3 \geq 0$

$y \geq -3$

$\therefore$ Range= $\{y: y \geq -3\}$

Exercise 2

1. For each of the following functions, state the domain and range

a. f(x) = 2x + 7 for 2 £ x £ 5

b. f(x) = x – 1 for -4 £ x £ 6

c. f(x) = 5 - 3x such that -2 £ f(x) < 8

2. for each of the following functions state the domain and range

a. $f(x) = x^2$

b. $f(x) = x^2+2$

c. f(x) = 2x + 1

d. $f(x) = 1 – x^2$

Exercise 3

1.The range of the function

f(x) 3 – 2x for 0 ³0 x £7 is;

a. y: -18£ y £3

b. y: -3£ y £18

c. y: 3 £y £18

d. y: -18 £ y £-3

2. The range of the function

f(x)=2x+1 is y: -3£ y £17 what is the domain of this function?

a. x**:** - 3£ x £17

b. x**:** - 2£ x £8

c. x: -17 £ x £3

3.Which of the following relations represents a function:

a. R = (x, y) : y = for x ≥0

b. R= (x, y) : y^2 = x-2 for x ≥0

c. R= (x, y) : y = for x ≥0 and y ≥0

d. R= (x, y) : x = 7 for all values of y

4.Which of the following relations is a function:

a. R = (x, y): -2 £ x £6, 3 £ y<8 and="" are="" both="" integers="" li="" where="" x="" y="">

b. R= (x, y): -2 £ x £6, 3 £ y<8 and="" are="" both="" integers="" li="" where="" x="" y="">

c. R= (x,y): y = √(x+2) for x ≥-2.

d. R = (x, y): y=√(2-x) for x ≤2 and y ≤0

5.Let f (x) = x^2 + 1. Which of the following is true?

a. f (-2) < f (0)

b. f (3)> f (-4)

c. f (-5) = f (5)

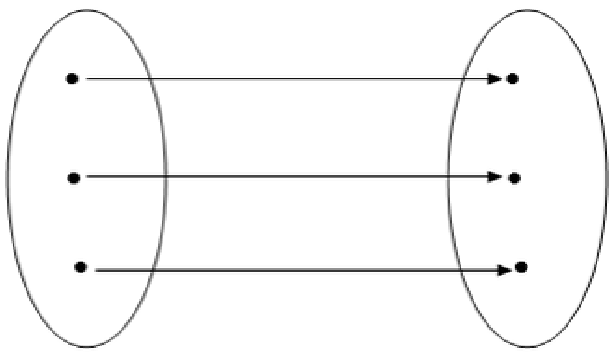
d. The function crosses , y – axis at 1

One to one and many to one functions:

One to functions;

A one to one function is a function in which one element from the domain is mapped to exactly one element in the range:

That is if a $\neq$b then f (a) $\neq$f (b)

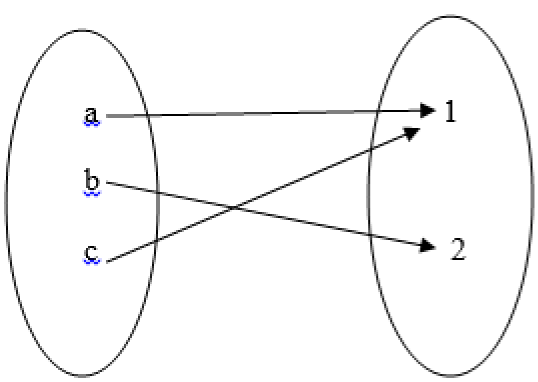


One to one function

Many to one function;

This is another type of function with a property that two or more elements from the domain can have one image (the same image).

i.e f (a) = f (b) but a$\neq$b



Examples of one to one functions

1. f (x) = 3x + 2

2. f (x) = x + 6

3. f (x) = x^3 + 1 etc

Examples of many to one function

1. f(x) = x^2 +1

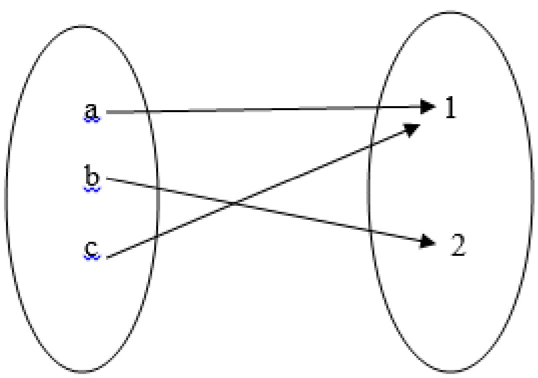
2. f(x) = x^4 – 2 etc

NB. All functions with odd degrees are one to one function and all functions with even degrees are many to one functions.

Example 5

Let A = -2, -1, 0, 1, 2 and B = 0, 1, 4 and the function f mapping each element from set A to those of B is defined as f(x)=x^2.Is f one to one function?

i.e f (a) = f (b) but a≠b



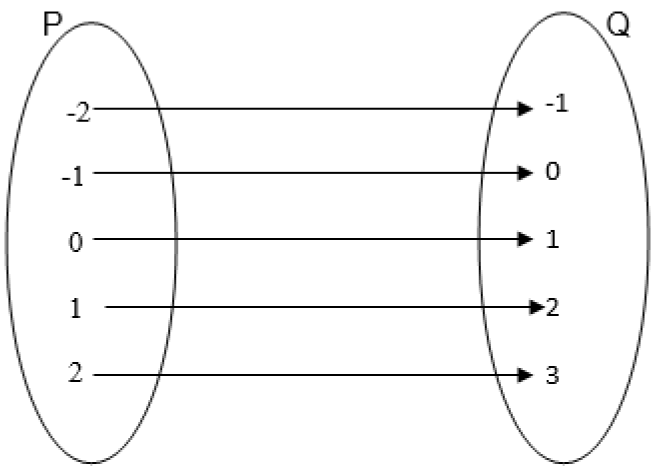
Example 6

Let P = {-2, -1, 0,1,2} and

Q = {-1, 0, 1, 2, 3}

g(x) = x + 1, is g one to one function?

Solution:



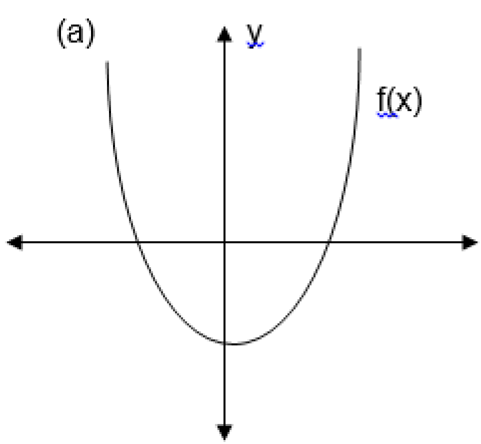
g (x) is one to one function because every element in P has only one image in Q

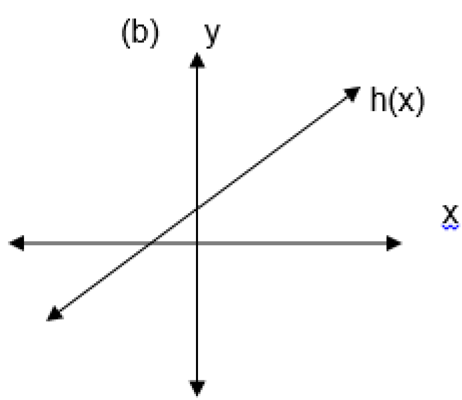
NB: In example 1, f(x) is not a one to one function because -2 and 2 in A have the same image in B, that is 4 is the image of both 2 and -2.

Also 1 is the image of both 1 nd -1.

Example 7

State whether or not if the following graphs represent a one to one function:

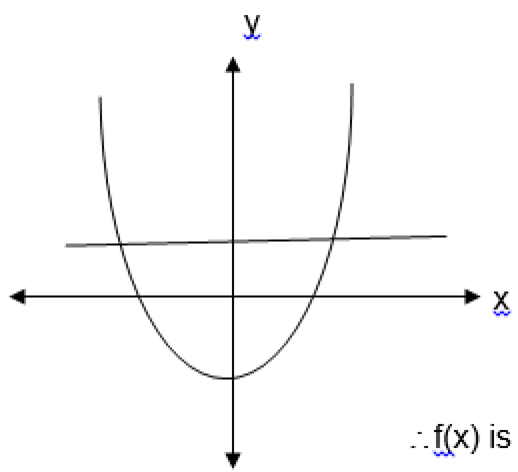




Solution:

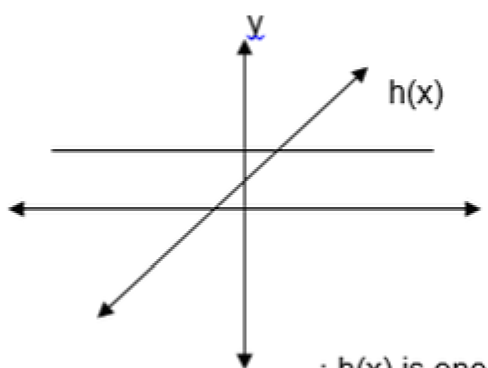
Draw a line parallel to the x axis and see if it crosses the graph at more than one points. If it does, then, the function is many to one and if it crosses at only one point then the graph represents a one to one function.

(a)



∴f(x) is many to one.

(b)



∴h(x) is one to one.

Graphic Function

Graphs of Functions

Draw graphs of functions

Many functions are given as equations, this being the case, drawing a graph of the equation is obtaining the graph of the equation which defines the function.

Note that, you can draw a graph of a function if you know the limits of its independent variables as well as dependent variables. i.e you must know the domain and range of the given function.

Example 8

Draw the graph of the following functions

a. $f(x) = 3x - 1$

b. $g(x) = x^2 - 2x - 1$

c. $h(x) = x^3$

Solution

$f(x) = 3x - 1$

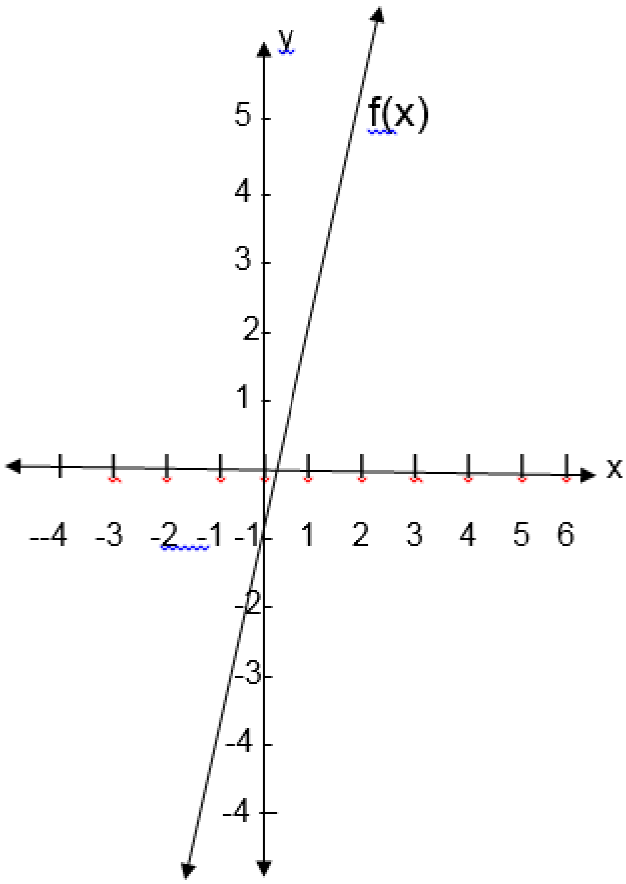
The domain and range of f are the sets of all real numbers

$f(x) = y = 3x - 1$

So $y = 3x - 1$

Table of value :

X	-1	0	2	1/3
Y	-4	-1	5	0



$g(x) = x^2 -2x -1$

$y=x^2-2-1$

a=-1, b=-2 1 and c=-1

The turning point T.P = $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right) = \left(\frac{2}{2}, \frac{-4-4}{4}\right) = (1,-2)$

T. p = (1, -2)

Y-Intercept is -1

So (x, y) = (0, -1)

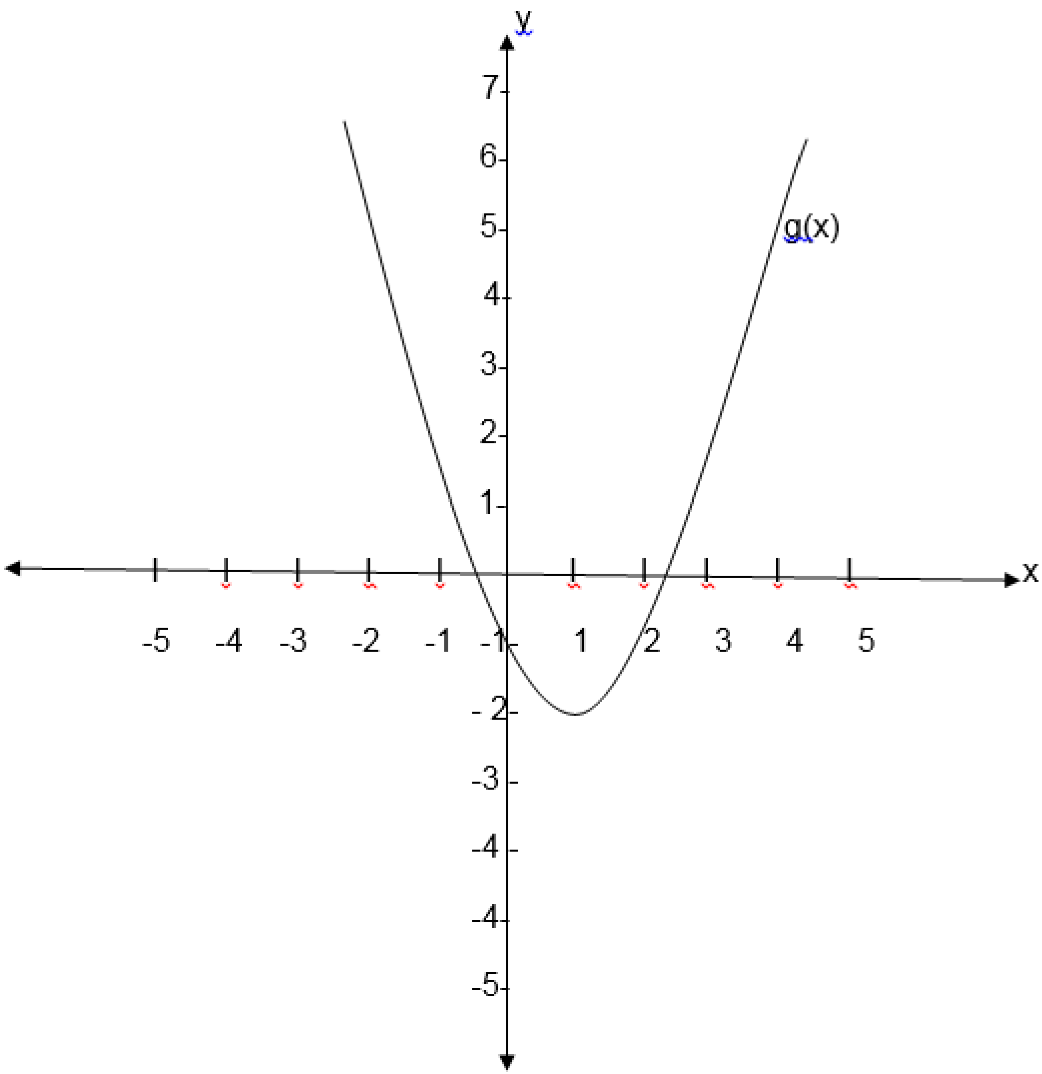
x – Intercepts are $(1 \pm \sqrt{2})$ which is obtained by solving for x when y = 0

So (x, y) = $[(1 \pm \sqrt{2}), 0]$

Other value are tabulated as follows

X	-1	1	3
Y	2	-2	2

Graph:



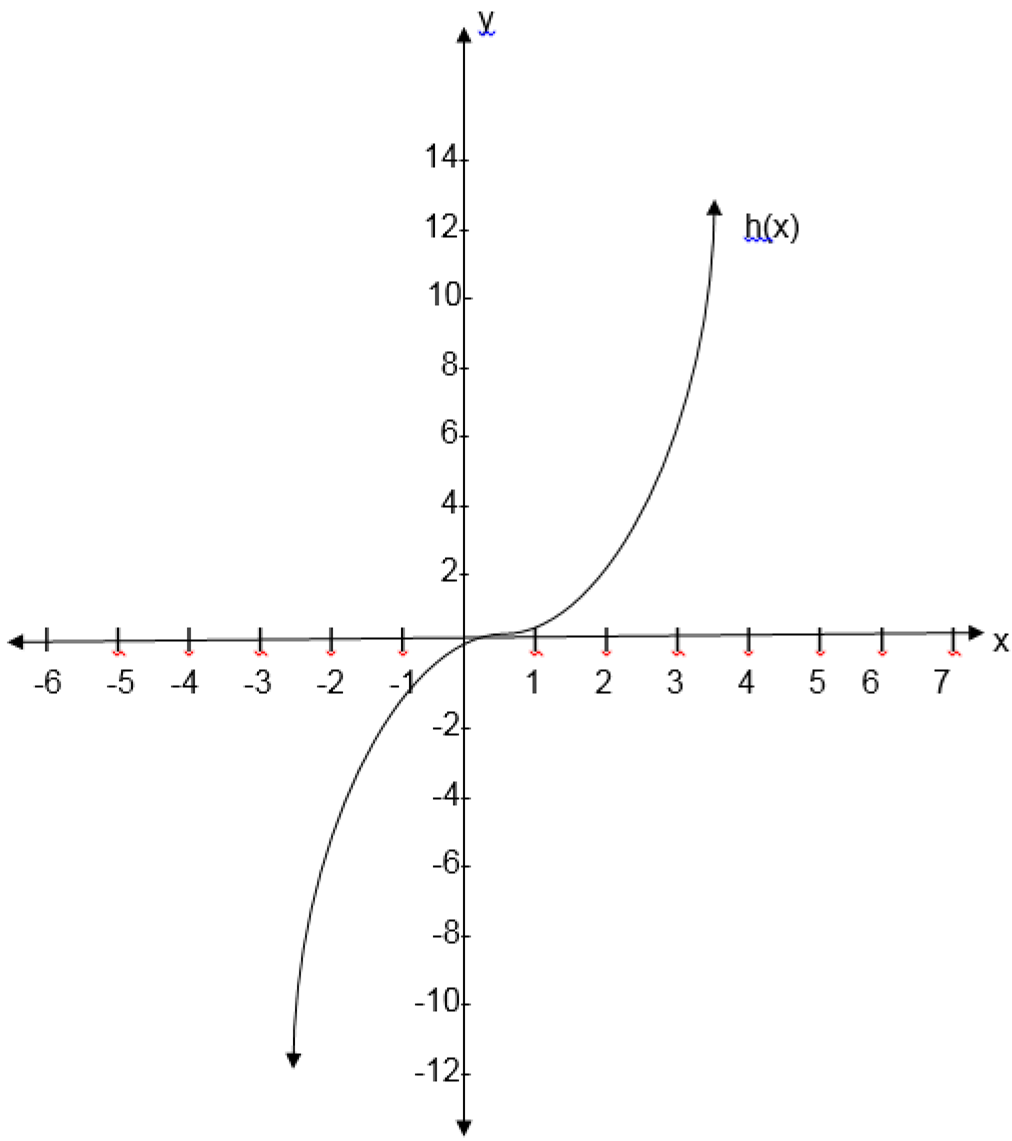
forh(x) = x^3

Solution:

Table of Values:

X	-2	-1	0-	1	2
H(x) =y	-8	-1	0	1	8

Graph:



The first graph is the graph of linear function, the second one is called the graph of a quadratic function and the last graph is for cubic function.

Example 9

Draw a graph of the function:

$f(x) = -1 + 6x-x^2$

Solution:

a=-1, b=6, c=-1

The turning point T.P = $\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right) = \left(\frac{-6}{-2}, \frac{4-36}{-4}\right) = (3,8)$

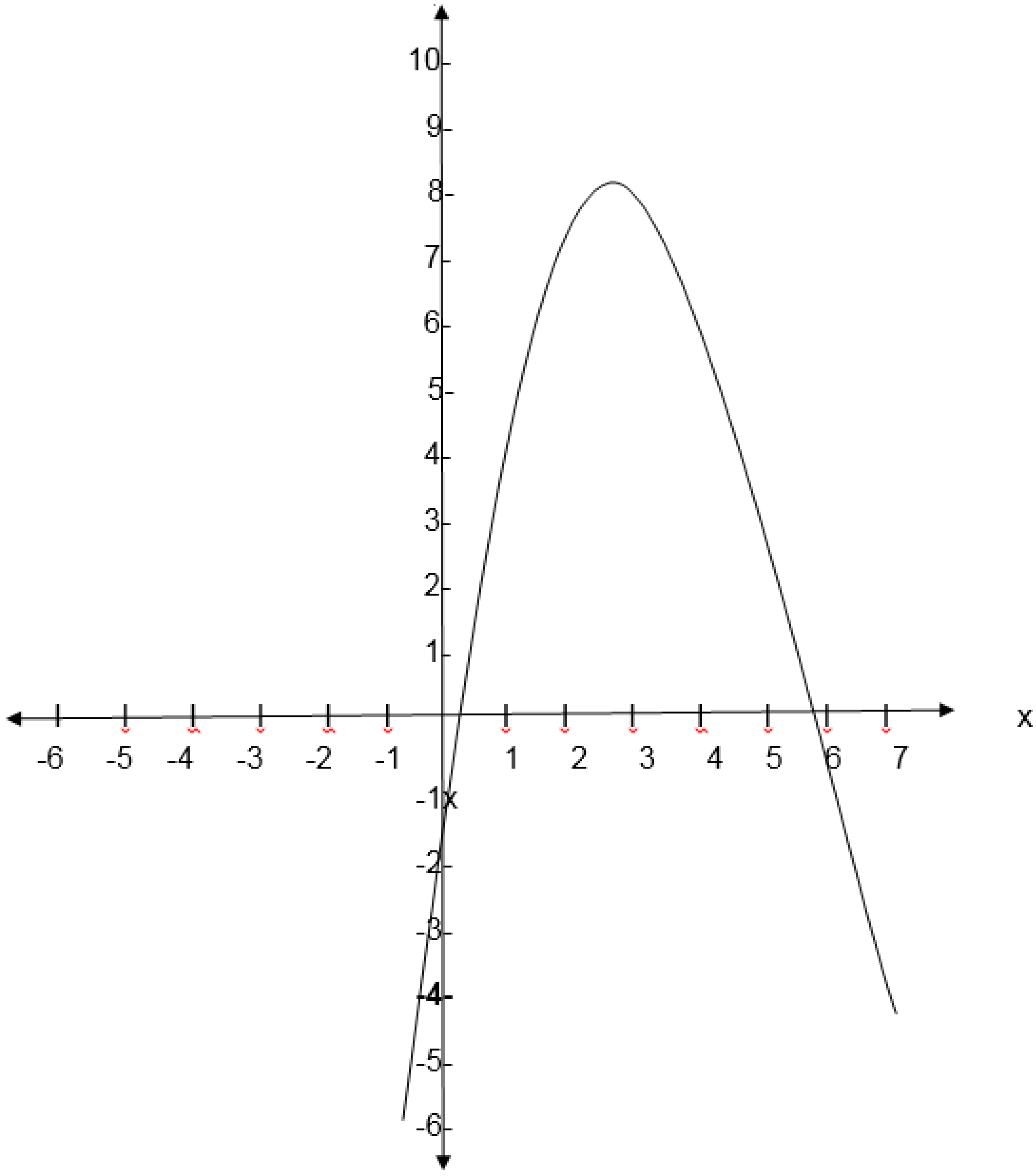
(X, y) = (3, 8)

Y-intercept = -1

(X, y)= (0,-1)

Other values

X	-4	-1	1	4	6	7
Y	-41	-8	4	7	-1	-8



Exercise 4

1.Which of the following are one to one function?

a. $f(x) = 3x - x^2$

b. $g(x) = x-1$

c. $k(x) = x^3+1$

d. $f(x) = x+x^2+x^3$

e. $k(x)=x^4$

2. Draw the graph of the following functions:

a. $f(x) = 3x – x2$

b. $h (x) = x+1$

c. $g(x) = x^3 - x^2 + 3$

3. At what values of x does the graph of the function $f(x) = x^2+x-6$ cross thex- axis?

a. x=-3 and x=7

b. x=8 and x=-6

c. x=-3 and x=2

d. x=4 and x=-1

4. Which of the following function is one to one function?

a. $f(x)=x^2+2$

b. $f(x) =x^4-x^2$

c. $f(x)=x^5-7$

d. $f(x)=x^2+x+2$

Functions with more than one part.

Some functions consist of more than one part. When drawing their graphs draw the parts separately.

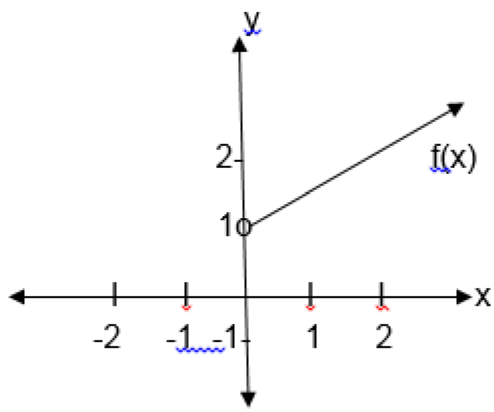
If the graph includes an end point, indicate it with a solid dot if it does not include the end point indicate it with a hollow dot.

E.g. draw the graphs of the functions

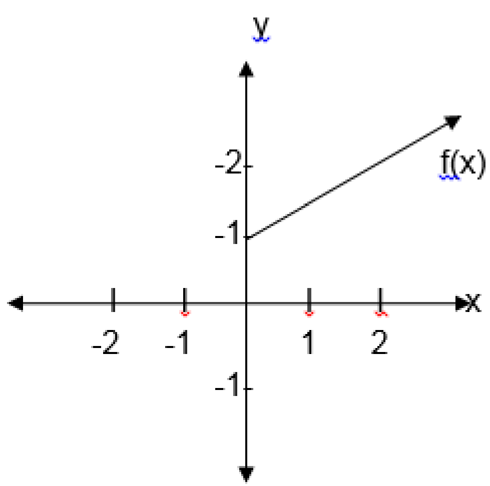
(a) F(x) x+1 for x>0

(b) f(x)=x+1for x³0

(a)



(b)



Example 10

Solved.

1. Let $f(x) = \begin{cases} -2 & \text{if} \quad x<0 \\ 0 & \text{if} \quad x=0 \\ 2 & \text{if} \quad x>0 \end{cases}$

(a)Find f(-8)

(b) Find f (16)

(c) Sketch its graph

(d) State the domain and range of f

Solution:

(a) f (-8 means f(x=-8)

But-8<0 and it is stated that when x<0

f(x) =-2

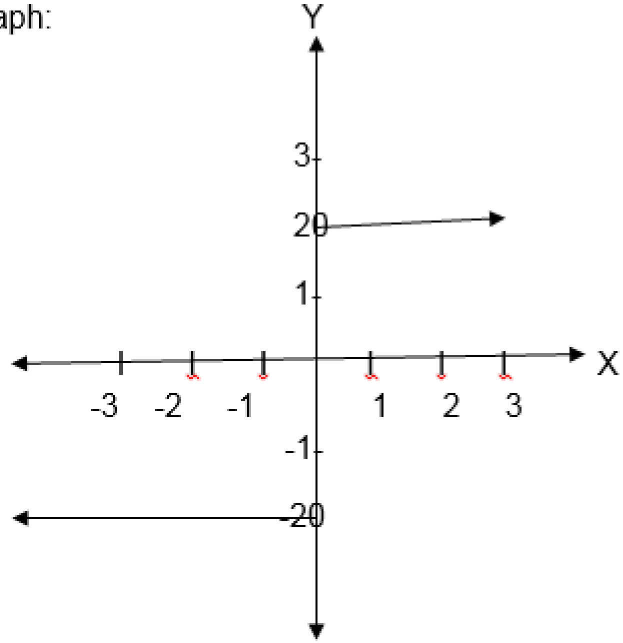
Therefore f (-8) =f(x<0) =-2

$\therefore$ F (-8) =-2

(b)Since $16>0$ and $f(x)=2$ if $x>0$, then $f(16)=2$

$\therefore f(16)=2$.

Graph:



(d)Domain = $\{$All read numbers$\}$

Range = $\{-2, 0, 2\}$

Exercise 5

Sketch the graph of each of the following functions and for each case state the domain and range.

(a) $F(x) \begin{cases} -1 \text{ if } & x<0 \\ 0 \text{ if } & x=0 \\ 2 \text{ if } & x>0 \end{cases}$

(b) $F(x) \begin{cases} x+1 \text{ if } & x \geq 1 \\ 2 \text{ if } & x+1<0 \end{cases}$

(c) $F(x) = \begin{cases} -3 & \text{for } x<0 \\ 1 & \text{for } 0=0 \leq x \leq 1 \\ 4 & \text{for } x>2 \end{cases}$

Absolute value functions (Modulus functions)

The absolute function is defined

As $f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

So if x is positive, $|x|$ remains unchanged and if x is negative, $|x|$ changes to –x which also becomes positive. Hence x is always grater or equal to Zero,

i.e. $|x| \geq 0$ for all real values of x.

E.g. x = -2, $|x| = |-2| = 2$ So x=-2 then $|x|=2$

And when x=2, $|x| = |2| = 2$

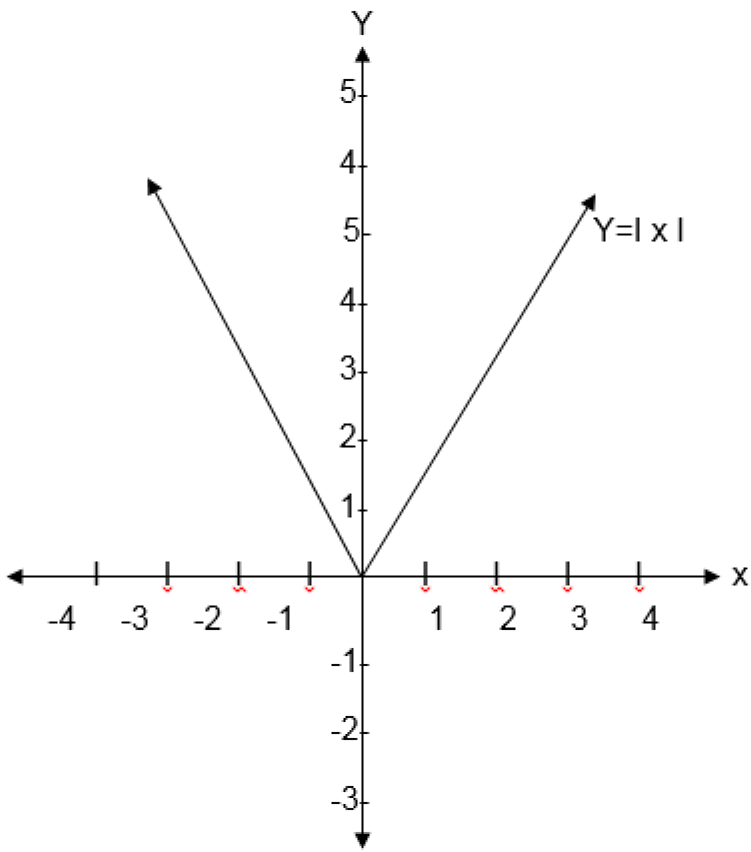
So x=2 then $|x|=2$

We can obtain the graph of f(x) $=|x|$ as follows;

Table of values

X	-3	-2	-1	0	1	2	3
I x I	3	2	1	0	1	2	3

Graph



Example 11

Solve the following

let f(x) =$|x + 2|$,

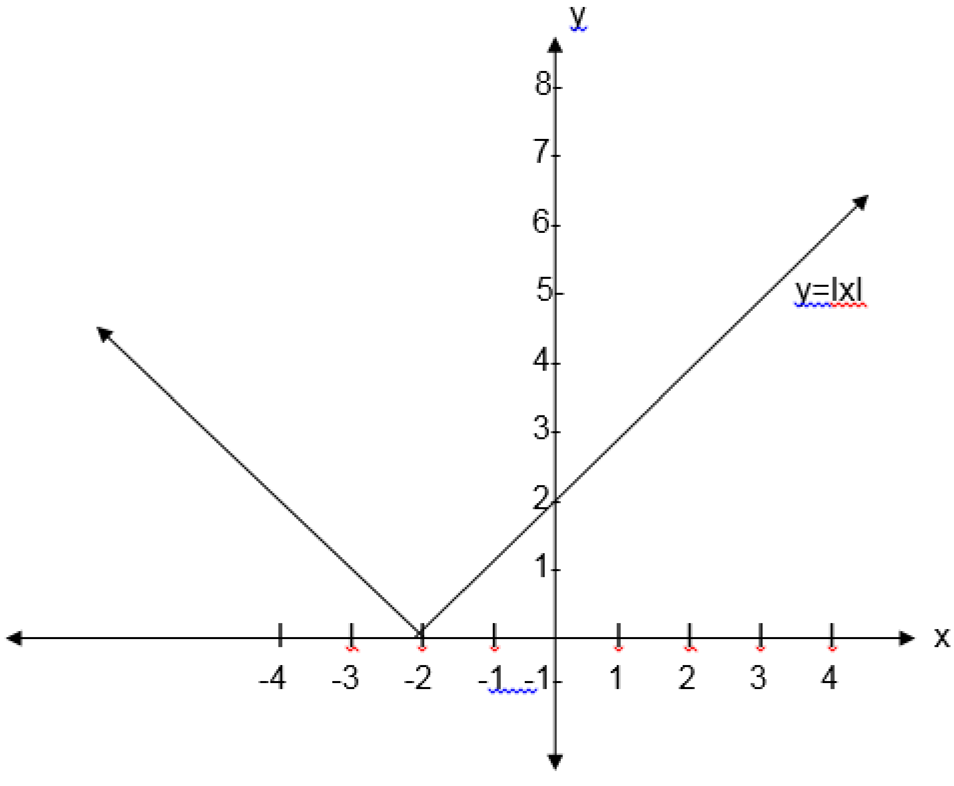
(a) sketch the graph of f(x)

(b) State its range.

Solution

table of values.

X	-4	-3	-2	0	1	4
y =f(x)	2	1	0	2	3	6



(b) Range = $\{y: y \geq 0\}$

When sketching a modulus function it is helpful to find first the minimum or maximum value of y.

Suppose the function is given as

y= a$|x - b|$+c

The vertex in this case is the point where x – b = 0 i.e. at x = b

The value of y becomes

y= ax0+ c = c

So (x, y) = (b, c) which is the vertex of the function

Step functions:

For any number x, $[x]$ is the value of x when rounded down to the integer below or equal to it.

So $[x]$ is the greatest integer which is less than or equal to x.

I.e. if n ≤ x ≤ n +1, then [x] = n

For example [1.5] = 1, [2.73] =2

[5, 3/8] =5, [6] =6, [-5] = -5

[-3.2] = -4 and [-2$\frac{2}{7}$] = -2

When drawing the graph of y = [x] the graph is horizontal between integers.

Example 12

Draw the graph of

$y = [x + ½]$

Solution

If $[x] = -4$, then $-4 \leq x < -3.5$

$[x] = -3$, then $-3 \leq x < -2.5$

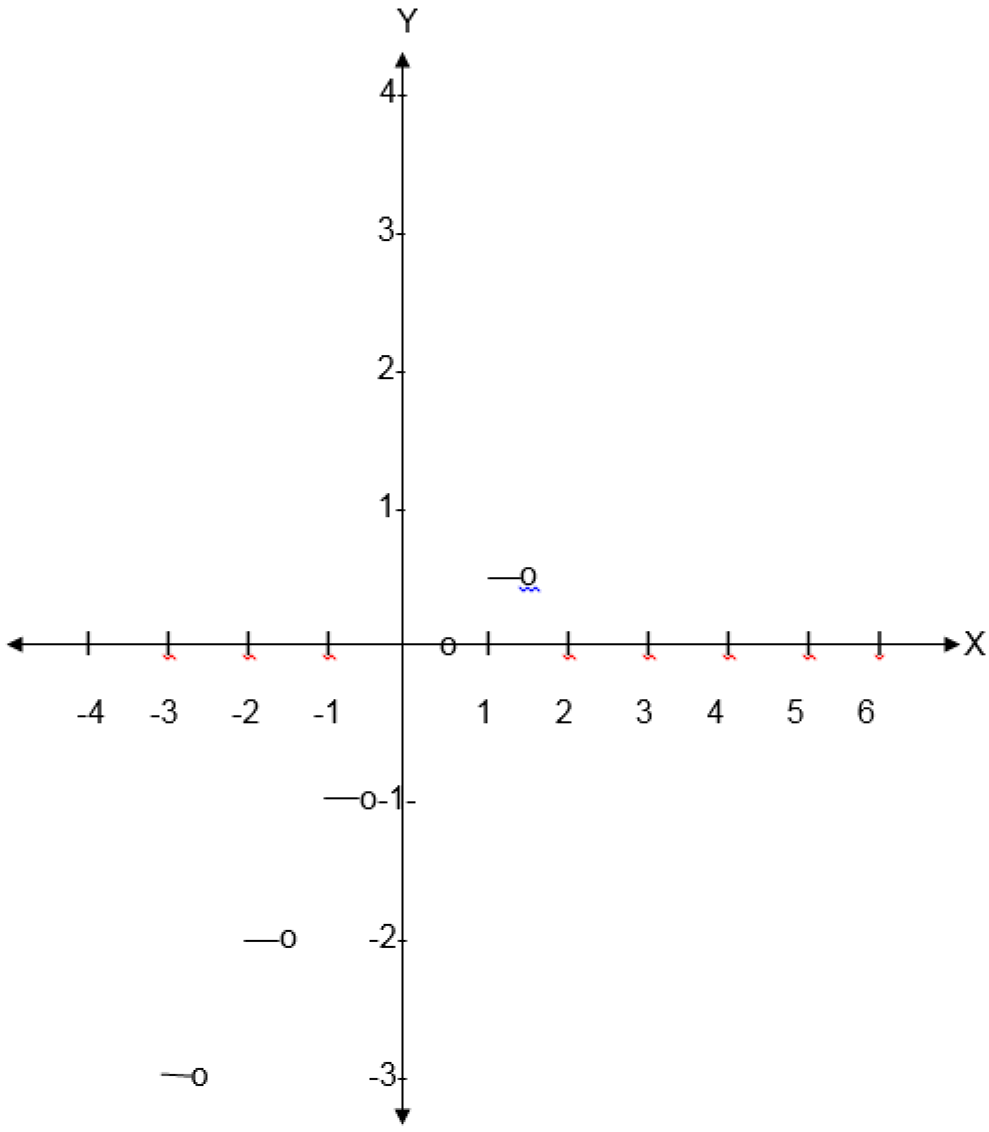
$[x] = -2$, then $-2 \leq x < -1.5$

$[x] = -1$, then $-1 \leq x < -0.5$

$[x] = 0$, then $0 \leq x < 0.5$

$[x] = 1$, then $1 \leq x < 1.5$

Graph



Note that the graph obtained is like steps such functions are called steps functions

Exercise 6

1. Draw the graph of

(a) F(x) =$|x+2|$+3 (b) f(x) =2-$|x|$

2. Draw the graph of f (x) = 3 |x -2 |+ 5 hence state the domain and range of f(x)

3. Given that f(x) = [x]

Find (a) f (-6.5) (b) f (12.01)

4. F(x) = 4–[x]

Find (a) f (-2.6) (b) f (3.3)

5. Draw the graph of the function f(x) = [x] + 3 and state its domain and range

6. Sketch the graph of the functions (a) y = [2x] (b) y= [x–3]

Inverse of a Function

The Inverse of a Function

Explain the inverse of a function

In the discussion about relation we defined the inverse of relation.

It is true that the inverse of the relation is also a relation.

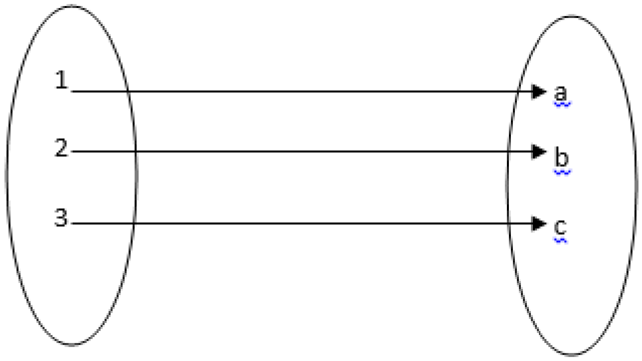
Similarly because a function is also relation then every function has its inverse

The Inverse of a Function Pictorially

Show the inverse of a function pictorially

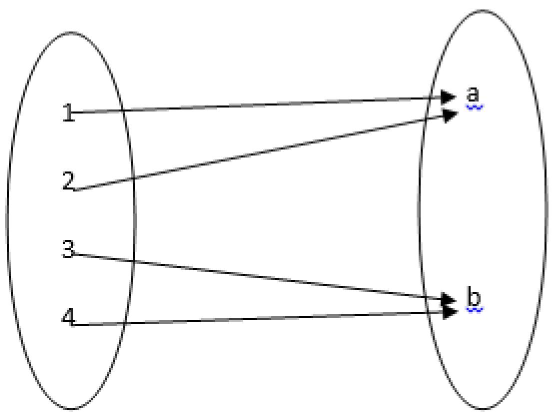
According to the definition of function the inverse of a function is also a function if and only if the function is one to one

(a)



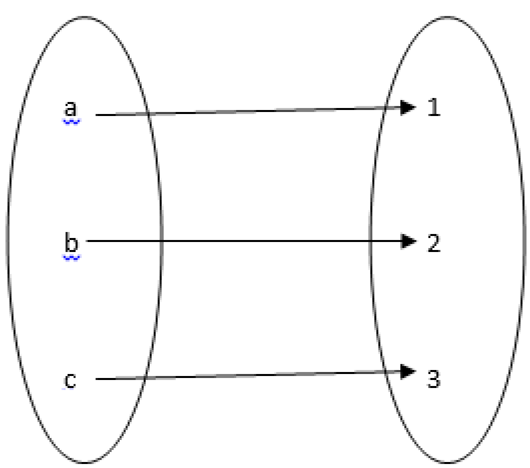
The inverse is also a function

(b)



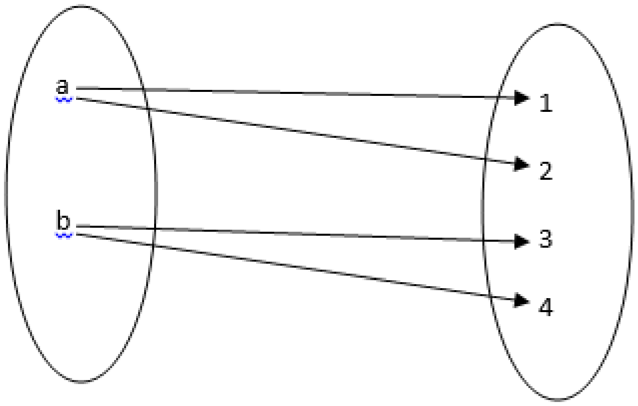
The inverse is not a function.

The inverse of a function given in (a) is shown below;



Its inverse is also a function

And the inverse of the function given in (b) is



The inverse is NOT a function.

The Inverse of a Function

Find the inverse of a function

If the function f is one to one function given by an equation, then its inverse is denoted by f^{-1} which is obtained by inter changing the variables x and y then making y the subject of the formula.

I.e. If y=f(x), then x = f^{-1} (y)

Example 13

1. Find the inverse of each of the following functions;

a. F(x) = 3x-6

b. F(x) =x^3

Solution:

(a) $f(x) = 3x-6$

Inverse: $y = 3x-6$

Then $x = 3y-6$

$x+6 = 3y$

$y=\frac{x+6}{3}$

$\therefore\ f^{-1}(x) = \frac{x+6}{3}$

(b) $f(x) = y = x^3$

$y = x^3$

Inverse: $x = y^3$

Then $y=\sqrt[3]{x}$

$\therefore f^{-1}(x)= \sqrt[3]{x}$

A Graph of the Inverse of a Function

Draw a graph of the inverse of a function

Example 14

find the inverse of the function $f(x) = x-5$ and then sketch the graph of $f^{-1}(x)$, also state the domain and range of $f^{-1}(x)$.

solution:

$F(x) = y = x-5$

$y=x-5$

Inverse: $x=y-5$

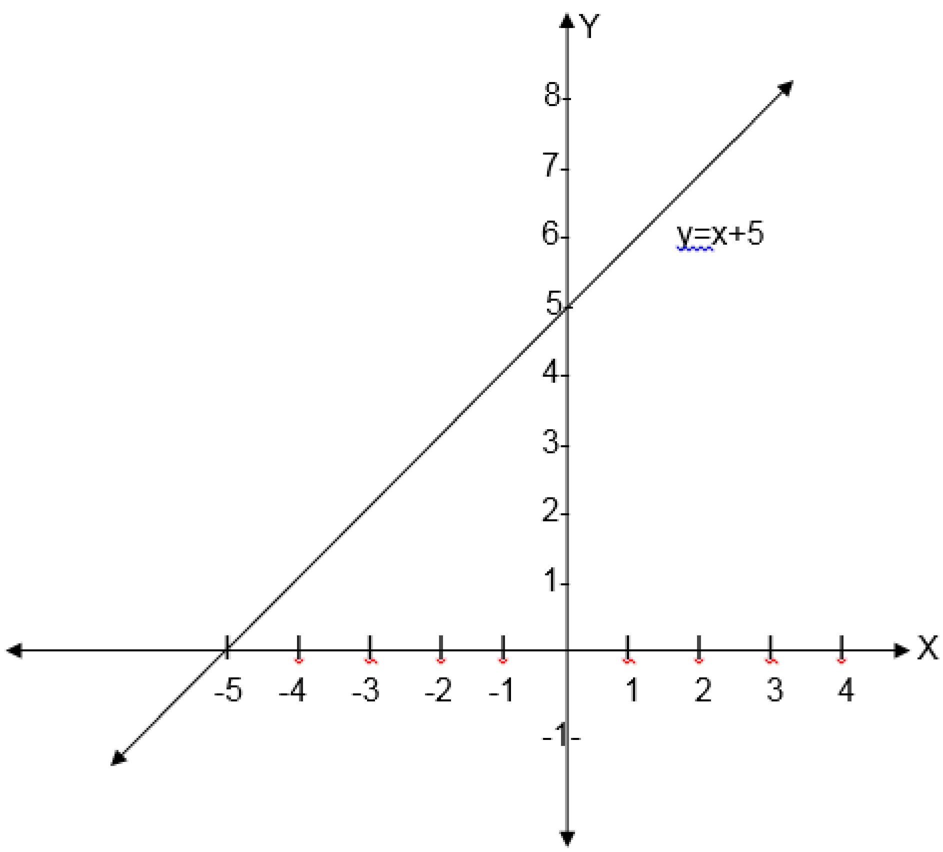
$y=x+5$

$\therefore f^{-1}(x) = x+5$

Graph of f-1 (x):

Table of values

X	0	-5
Y	5	0



Domain = {All real numbers}

Range = {All real numbers}

NB: if a function f takes a domain A to a range B, then the inverse f^{-1} takes B back to A.

Hence the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

The Domain and Range of Inverse of Functions

State the domain and range of inverse of functions

Example 15

Solve;

Let f(x) = 3x -2 for $\{0 \leq x \leq 5\}$

Where x is a real number.

Find the domain and range of f^{-1}

Solutions:

The function takes all values of y between

f (0) =-2 and f(5)=13

So range = $\{y: -2 \leq y \leq 13\}$

The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}

$\therefore$ The domain of $f^{-1} = \{-2 \leq x \leq 13\}$

And range = $\{0 \leq y \leq 5\}$

Exercise 7

1.Find the inverse of each of the following functions:

(a) $F(x) = 3x^2+8$

(b) $F(x) = x-9$

2. given that $f(x) = 7x-4$ find $f^{-1}(8)$

3. given that $f(x) = \frac{3x+4}{5}$ for $-3 \le x \le 8$

Find $f^{-1}(x)$ and state the domain and range of $f^{-1}(x)$

4. Let $f(x) = x^3-1$

Evaluate (1) $f^{-1}(6)$ (ii) $f^{-1}(k)$

Exercise 8

1. given that $f(x) = x^2-2^{[x]}+3$, what is the value of f (-4)?

(a) 15 (b) 11 (c) 27 (d)-5 ( )

2. Let $f(x) = \begin{cases} x, & if\ x \le 0 \\ x-2, & if\ x > 0 \end{cases}$ What is a value of f(6)?

(a) 4 (b) 6 (c)-8 (d)8 ( )

3. given that $f(x) = |x-5|-7$, what is the minimum value of f(x)?
(a) 7 (b) 0 (c) 5 (d) -7 ( )

4. Let f(x) =2x –i/3 what is the value of x such that f-1 (x) =2?

(a) $\frac{4}{3}$ (b)$\frac{3}{2}$ (c) $\frac{11}{3}$ (d) $\frac{8}{3}$ ( )

5.The function whose inverse is $f^{-1}(x) = 3x+5$ is given by;

(a) $f(x) = \frac{1}{3}(x-5)$ (b) $f(x) = \frac{3}{5x}$ ( )
(b) (c) $f(x) = \frac{1}{5}(x+3)$ (d) $f(x) = 5x-3$